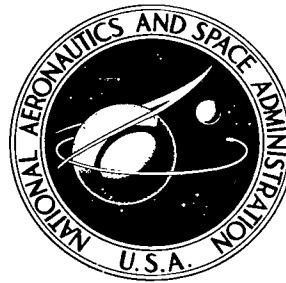
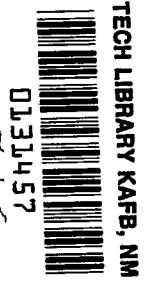


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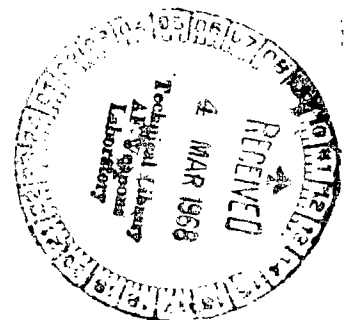
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ELASTIC-PLASTIC TORSION PROBLEM FOR STRAIN-HARDENING MATERIALS

by Alexander Mendelson

Lewis Research Center

Cleveland, Ohio





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STRAIN-HARDENING MATERIALS

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SUMMARY

A simple and straightforward procedure is presented for solving the elastic-plastic problem for the torsion of a solid prismatic bar made of a strain-hardening material. The procedure is based on the method of "successive elastic solutions" or successive approximations. For circular cross sections, the problem reduces to the solution of a nonlinear algebraic equation. For the case of a circular cross section with linear strain hardening, the solution is obtained in closed form. Results are presented for bars of rectangular and circular cross sections with linear strain hardening.

INTRODUCTION

Although the problem of elastic torsion of prismatic bars is one of the classical problems of mechanics and has received extensive treatment, the corresponding problem of elastic-plastic torsion has as yet not been satisfactorily attacked.

The usual solutions to the plastic torsion problem assume perfectly plastic materials and are generally limited to cases in which the complete cross section is plastic, because the elastic-plastic boundary is considered difficult to find. Furthermore, for completely plastic sections of bars made of perfectly plastic materials, the ingenious sand-hill analogy of Nádai (ref. 1) enables one to obtain solutions experimentally. Examples of solutions of this type can be found in references 1 to 5. An example of a solution for a strain-hardening material can be found in reference 6 for a special type of stress-strain law.

This report presents a relatively simple solution to the elastic-plastic torsion problem for strain-hardening materials. Use is made of the method of successive elastic solutions for successive approximations which have been applied so successfully to many other types of problems as described in references 7 to 10, among others. For the

practical and important case of a bar of circular cross section, the problem reduces to the solution of a nonlinear algebraic equation. For linear strain hardening, the solution is obtained in closed form.

METHOD OF ANALYSIS

Basic Equations

The analysis begins with the Saint-Venant semi-inverse method as for the elastic case. Consider a prismatic bar subjected to a twisting couple as shown in figure 1. One

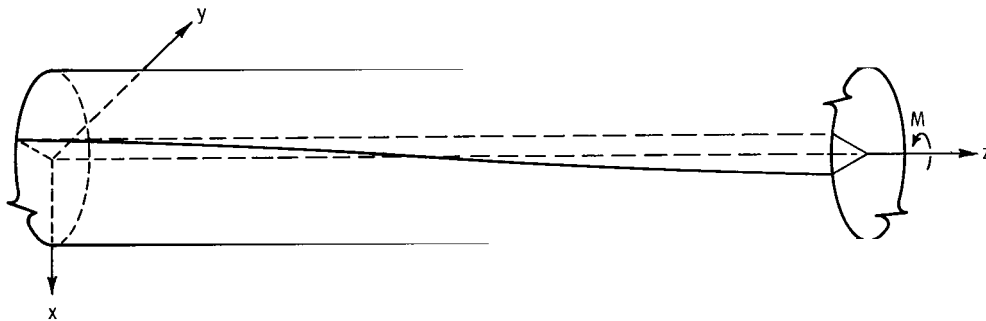


Figure 1. - Prismatic bar subject to twisting couple.

end of the bar is assumed fixed against rotation but not against warping, and, at the other end, a couple M with moment along the z -axis is applied. Following Saint-Venant, it is assumed that the displacements in the x , y , and z directions, respectively, are given by

$$\left. \begin{aligned} u &= -yz\alpha \\ v &= xz\alpha \\ w &= w(x, y, \alpha) \end{aligned} \right\} \quad (1)$$

where α is the angle of twist per unit length. (All symbols are defined in appendix A.) The warping function w is directly proportional to the angle of twist per unit length α in the elastic case. For the plasticity problem this is, in general, no longer true.

Substituting equations (1) into the usual strain-displacement relations results in

$$\left. \begin{aligned} \gamma_{xz} &= -\alpha y + \frac{\partial w}{\partial x} \\ \gamma_{yz} &= \alpha x + \frac{\partial w}{\partial y} \end{aligned} \right\} \quad (2)$$

all the other strains being zero. If equations (2) are now substituted into Saint-Venant's six compatibility equations, four of them are identically satisfied and the remaining two become

$$\left. \begin{aligned} \frac{\partial}{\partial x} \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} \right) &= 0 \\ \frac{\partial}{\partial y} \left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} \right) &= 0 \end{aligned} \right\} \quad (3)$$

which gives

$$\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} = \text{constant} \quad (4)$$

From equations (2), it follows that the constant appearing in equation (4) must equal 2α . Hence,

$$\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} = 2\alpha \quad (5)$$

Equation (5) is the compatibility equation for this problem.

The only equilibrium equation not satisfied identically is

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \quad (6)$$

The stress-strain relations can be written

$$\left. \begin{aligned} \gamma_{xz} &= \frac{1}{G} \tau_{xz} + \gamma_{xz}^p \\ \gamma_{yz} &= \frac{1}{G} \tau_{yz} + \gamma_{yz}^p \end{aligned} \right\} \quad (7)$$

where γ_{xz}^p and γ_{yz}^p are the total accumulated plastic shear strains. Substituting equation (7) into equation (5) yields the compatibility equation in terms of the stresses

$$\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y} = 2G\alpha + g(x, y) \quad (8)$$

where

$$g(x, y) = G \left(\frac{\partial \gamma_{xz}^p}{\partial y} - \frac{\partial \gamma_{yz}^p}{\partial x} \right) \quad (9)$$

The stress function φ is now introduced such that

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \quad \tau_{yz} = - \frac{\partial \varphi}{\partial x} \quad (10)$$

Then the equilibrium equation (6) is identically satisfied, and the compatibility equation (8) becomes

$$\nabla^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = - 2G\alpha - g(x, y) \quad (11)$$

For the elastic problem $g(x, y)$ is equal to zero.

The boundary condition for this problem can be obtained directly from the condition that the lateral surface is force free. As shown in any standard text this becomes simply

$$\varphi = 0 \quad (12)$$

on the boundary, provided that the section is simply connected. For multiply-connected cross sections, the stress function φ is equal to a different constant on each boundary. We will concern ourselves here only with simply connected sections, that is, solid bars.

From equation (10) it follows that the resultant shear stress is equal to the gradient of φ ; that is,

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \sqrt{\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2} = |\text{grad } \varphi| \quad (13)$$

The resultant moment can readily be found by integration to be

$$M = 2 \iint \varphi \, dx \, dy \quad (14)$$

The elastic-plastic torsion problem has thus been reduced to solving equation (11), subject to boundary condition (12). For the elasticity problem, the right side of equation (11) is a constant, and solutions, at least for simple shapes, can readily be obtained. For the plasticity problem, the right side of equation (11) is a function of the plastic strains and is therefore unknown until the solution is obtained. To solve the problem, use is made of the method of successive elastic solutions or successive approximations.

Before proceeding with the details of the solution, the following should be noted: As shown in reference 11, both the total and incremental theories of plasticity furnish the same solution to the torsion problem provided either (1) the cross section is circular, or (2) the material is perfectly plastic. It is reasonable to assume, therefore, that this will be approximately true for most practical problems. Indeed, it has been shown in reference 6 that for the case of a square cross section with strain hardening there is little difference between total and incremental theories. In what follows, therefore, use will be made of the total or deformation theories of plasticity and the load will be assumed to be applied in one step. The use of incremental theories does not appreciably complicate the problem, and the necessary formulation is given in appendix B for those who desire to use it.

The deformation theory of plasticity using the von Mises yield criterion results in the following plastic strain - total strain relations as shown in reference 7.

$$\left. \begin{aligned} \gamma_{xz}^p &= \frac{\epsilon_p}{\epsilon_t} \gamma_{xz} \\ \gamma_{yz}^p &= \frac{\epsilon_p}{\epsilon_t} \gamma_{yz} \end{aligned} \right\} \quad (15)$$

where

$$\left. \begin{aligned} \epsilon_t &= \sqrt{\frac{\gamma_{xz}^2 + \gamma_{yz}^2}{3}} \\ \epsilon_p &= \sqrt{\frac{(\gamma_{xz}^p)^2 + (\gamma_{yz}^p)^2}{3}} \end{aligned} \right\} \quad (16)$$

and the equivalent plastic strain ϵ_p is related to the equivalent total strain ϵ_t through the stress-strain curve of the material and can be written, in general, as

$$\epsilon_p = f(\epsilon_t) \quad (17)$$

These plasticity relations will now be used for obtaining a solution by successive approximations.

Method of Solution

It is convenient to introduce the following dimensionless quantities:

$$\left. \begin{aligned} U &\equiv \frac{\varphi}{2G\epsilon_0 a} & \beta &\equiv \frac{a\alpha}{\epsilon_0} & M^* &\equiv \frac{M}{2G\epsilon_0 a^3} \\ \gamma_x &\equiv \frac{\gamma_{xz}}{2\epsilon_0} & \gamma_y &\equiv \frac{\gamma_{yz}}{2\epsilon_0} & \gamma_x^p &\equiv \frac{\gamma_{xz}^p}{2\epsilon_0} & \gamma_y^p &\equiv \frac{\gamma_{yz}^p}{2\epsilon_0} \\ e_p &\equiv \frac{\epsilon_p}{\epsilon_0} & e_t &\equiv \frac{\epsilon_t}{\epsilon_0} \\ \tau_x &\equiv \frac{\tau_{xz}}{2G\epsilon_0} & \tau_y &= \frac{\tau_{yz}}{2G\epsilon_0} \\ \xi &\equiv \frac{x}{a} & \eta &= \frac{y}{a} \end{aligned} \right\} \quad (18)$$

where ϵ_0 and σ_0 are the yield strain and yield stress, respectively, in the uniaxial tensile test, related to each other by $\sigma_0 = E\epsilon_0$, and a is a characteristic dimension of the cross section.

The system of equations to be solved for a simply connected cross section can now be written as follows:

$$g(\xi, \eta) = \frac{\partial \gamma_x^p}{\partial \eta} - \frac{\partial \gamma_y^p}{\partial \xi} \quad (19a)$$

$$\left. \begin{aligned} \nabla^2 U &= -\beta - g(\xi, \eta) \\ U &= 0 \text{ on boundary} \end{aligned} \right\} \quad (19b)$$

$$\tau_x = \frac{\partial U}{\partial y} \quad \tau_y = -\frac{\partial U}{\partial \xi} \quad (19c)$$

$$\gamma_x = \tau_x + \gamma_x^p \quad \gamma_y = \tau_y + \gamma_y^p \quad (19d)$$

$$e_t = \frac{2}{\sqrt{3}} \sqrt{\gamma_x^2 + \gamma_y^2} \quad (19e)$$

$$e_p = F(e_t) \quad (19f)$$

$$\gamma_x^p = \frac{e_p}{e_t} \gamma_x \quad \gamma_y^p = \frac{e_p}{e_t} \gamma_y \quad (19g)$$

The successive approximation method proceeds in the following manner. The plastic strains are assumed to be zero everywhere. Equation (19b) is solved by any available method. The stresses, the total strains, and equivalent total and plastic strains are computed by means of equations (19c) to (19f) with the help of the stress-strain curve. If, at any point in the cross section, the equivalent plastic strain as computed from equation (19f) is negative, this point is in the elastic region, and the plastic strains at this point are set equal to zero. Otherwise, new approximations to the plastic strains are calculated by means of equations (19g). One then returns to equation (19a), and the process is repeated until the difference between two successive iterations differs by less than some preassigned value. For all the cases treated herein, convergence was obtained very rapidly. A discussion of the convergence of this technique for other types of problems can be found in references 7 and 8. The method will be illustrated for bars of rectangular and circular cross sections. For a circular cross section with linear strain hardening, the solution can be obtained in closed form.

Bar With Rectangular Cross Section

Equation (19b) is the well-known Poisson equation of mathematical physics which can be solved by a variety of methods. Herein the finite-difference method is used because it is simple and straightforward and is readily programmed for a digital computer.

Consider a bar of rectangular cross section as shown in figure 2. Note first that, because of symmetry, only one quadrant of the section need be considered. For a square cross section, the diagonals are also lines of symmetry, and only one octant is used. The quadrant is divided into a grid of $n \times m$ points as shown in figure 3. At each of the grid points such as the point designated by (i, j) equation (19b) is written in finite difference form

$$U_{i+1,j} + U_{i-1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j} = -(\beta + g_{i,j})h^2 \quad (20)$$

where h is the constant grid spacing divided by the half width of the plate a and is the same in both the x and y directions, and

$$g_{i,j} = \frac{1}{2h} \left(\gamma_{x,i,j+1}^p - \gamma_{x,i,j-1}^p - \gamma_{y,i+1,j}^p + \gamma_{y,i-1,j}^p \right) \quad (21)$$

An equation such as equation (20) can be written for each of the mn grid points resulting in a set of mn simultaneous linear equations for the unknown values of U at each of the points. Actually, the number of equations to be solved is $(m-1)(n-1)$ rather than mn , since the boundary conditions require U to be zero at the upper and

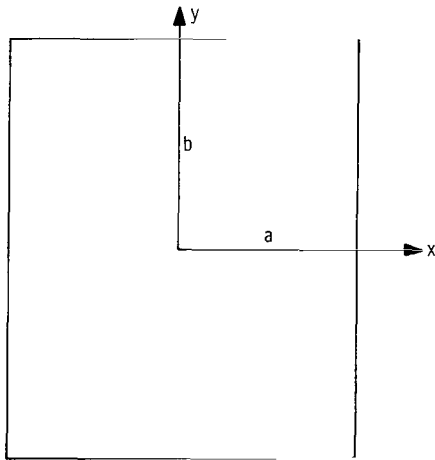


Figure 2. - Rectangular cross section.

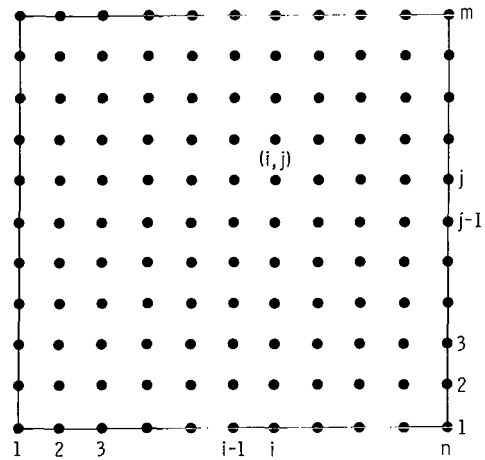


Figure 3. - Finite-difference grid for one quadrant of rectangle.

right hand boundaries of the quadrant. For a square section, only the points on the diagonal and to the right of the diagonal (or left) need be considered, resulting in $1/2n(n - 1)$ equations. Along the lower boundary, because of symmetry, equation (20) becomes

$$U_{i+1,1} + U_{i-1,1} + 2U_{i,2} - 4U_{i,1} = -(\beta + g_{i,1})h^2$$

Along the left boundary,

$$U_{1,j-1} + U_{1,j+1} + 2U_{2,j} - 4U_{1,j} = -(\beta + g_{1,j})h^2$$

at the center,

$$4U_{2,1} - 4U_{11} = -(\beta + g_{11})h^2$$

and along the diagonal of a square,

$$2U_{i+1,i} + 2U_{i,i-1} - 4U_{i,i} = -(\beta + g_{i,i})h^2$$

Once the values of U are determined at all the grid points, corresponding values of plastic strains are computed by means of equations (19c) to (19g). The $g_{i,j}$ are then recomputed, and equations (19b) are solved again. The process is repeated until convergence is obtained. An example using this technique is described in the section RESULTS AND DISCUSSION.

Bar With Circular Cross Section

For a bar with a circular cross section, the solution is greatly simplified. In particular, for the case of linear strain hardening, a closed-form solution can be obtained. In polar coordinates, the displacements are

$$\left. \begin{array}{l} u_r = 0 \\ u_\theta = \alpha r z \end{array} \right\} \quad (22)$$

and the only nonzero strain is

$$\gamma_{\theta z} = \alpha r \quad (23)$$

The stress-strain relation can therefore be written

$$\frac{1}{G} \tau_{\theta z} = \alpha r - \gamma_{\theta z}^p \quad (24)$$

The von Mises equivalent stress reduces to

$$\sigma_e = \sqrt{3} \tau_{\theta z} = \sqrt{3} G \alpha r - \sqrt{3} G \gamma_{\theta z}^p \quad (25)$$

and the equivalent plastic strain is

$$\epsilon_p = \frac{1}{\sqrt{3}} \gamma_{\theta z}^p \quad (26)$$

Hence,

$$\sigma_e = \sqrt{3} G \alpha r - 3 G \epsilon_p \quad (27)$$

Let

$$\left. \begin{array}{l} \frac{\sigma_e}{2G\epsilon_0} \equiv S_e \quad \frac{r}{a} \equiv \rho \\ \frac{\alpha a}{\epsilon_0} \equiv \beta \quad e_p \equiv \frac{\epsilon_p}{\epsilon_0} \quad e_\theta^p \equiv \frac{\gamma_{\theta z}^p}{2\epsilon_0} \end{array} \right\} \quad (28)$$

where a is the radius of the bar. Then equation (27) can be written in dimensionless form as

$$S_e = \frac{\sqrt{3}}{2} \beta \rho - \frac{3}{2} e_p \quad (29)$$

The stress-strain curve equation can be written in dimensionless form as

$$S_e = f(e_p) \quad (30)$$

and combining with equation (29) results in

$$2f(e_p) = \sqrt{3}\beta\rho - 3e_p \quad (31)$$

which can be solved iteratively for e_p .

Equation (31) is valid only in the plastic region. Let this region extend between $\rho = \rho_c$ and $\rho = 1$. To determine the position of the elastic-plastic boundary ρ_c , let $\epsilon_\rho = 0$ when $\sigma_e = \sigma_0$ or when

$$S_e = 1 + \mu$$

Hence, from equation (29),

$$\rho_c = \frac{2(1 + \mu)}{\sqrt{3}\beta} \quad (32)$$

which depends only on Poisson's ratio μ and the yield strain but not on the stress-strain curve. The value of β at which plastic flow just starts is found from equation (32) by setting ρ_c equal to 1. Thus, the critical value of β will be

$$\beta_c = \frac{2(1 + \mu)}{\sqrt{3}} \quad (33)$$

and the critical angle of twist per unit length will be

$$\left. \begin{aligned} \alpha_c &= \frac{\epsilon_0}{a} \beta_c = \frac{2(1 + \mu)}{\sqrt{3}} \frac{\epsilon_0}{a} \\ \alpha_c G &= \frac{1}{\sqrt{3}} \frac{\sigma_0}{a} \end{aligned} \right\} \quad (34)$$

or

In summary, the strain-hardening solution is found as follows: The elastic-plastic boundary ρ_c is first determined from equation (32). The stress and strain in the elastic region for $\rho \leq \rho_c$ are then computed from equations (23) and (24) with $\gamma_{\theta z}^p$ set equal to zero. In the plastic region $\rho > \rho_c$ equation (31) is usually solved by an

iterative method. Then, $\gamma_{\theta z}^p$ can be computed from equation (26), and the shear stress from equation (24). Once the shear stress is known throughout the section, the torque can be computed by integration.

Let us now consider the case of linear strain-hardening. Equation (30) for the stress-strain curve, can be written as follows:

$$S_e = (1 + \mu) \left(1 + \frac{m}{1 - m} e_p \right) \quad (35)$$

(where the strain-hardening parameter m is the ratio of the slope of the linear hardening curve to the slope of the elastic curve). Hence, equation (31) becomes

$$2(1 + \mu) \left(1 + \frac{m}{1 - m} \rho_p \right) = \sqrt{3} \beta \rho - 3e_p$$

or

$$e_p = \frac{\sqrt{3} \beta \rho - 2(1 + \mu)}{3 + \frac{2(1 + \mu)m}{1 - m}} \quad \rho \geq \rho_c \quad (36)$$

Note that the critical value of ρ is obtained when the numerator of equation (36) vanishes, which results again in equation (32).

Once the equivalent plastic strain is known from equation (36), the plastic shear strain and the stress are computed from equations (26) and (24). Thus, a complete solution in closed form is obtained. To compute the torque, define

$$\tau_\theta \equiv \tau_{\theta z} / 2G\epsilon_0$$

Then,

$$M^* \equiv M / 2G\epsilon_0 a^3 = 2\pi \int_0^1 \tau_\theta \rho^2 d\rho \quad (37)$$

and substituting

$$\tau_\theta = \frac{1}{2} \beta \rho \quad \rho \leq \rho_c$$

$$\tau_\theta = \frac{1}{2} \beta \rho - e_\theta^p \quad \rho \geq \rho_c$$

results in

where

$$\left. \begin{aligned} M^* &= \frac{\pi\beta}{4} - \sqrt{3}\pi \left[\frac{1}{4} A (1 - \rho_c^4) + \frac{1}{3} B (1 - \rho_c^3) \right] \\ A &= \frac{\sqrt{3}\beta}{3 + \frac{2(1+\mu)m}{1-m}} \\ B &= - \frac{2(1+\mu)}{3 + \frac{2(1+\mu)m}{1-m}} \end{aligned} \right\} \quad (38)$$

Note that for $\rho_c = 1$ (no plastic flow) the torque reduces to the elastic torque as given in standard texts. For a perfectly plastic material

$$m = 0 \quad A = \frac{\beta}{\sqrt{3}} \quad B = -\frac{2}{3} (1 + \mu)$$

and

$$M^* = \frac{2\pi(1+\mu)}{3\sqrt{3}} \left(1 - \frac{1}{4} \rho_c^3 \right)$$

or

$$M = \frac{2\pi a^3 \sigma_0}{3\sqrt{3}} \left[1 - \frac{1}{12\sqrt{3}a^3} \left(\frac{\sigma_0}{G\alpha} \right)^3 \right] \quad (39)$$

and the classical solution as given in reference 2 is recovered.

RESULTS AND DISCUSSION

Calculations were performed by the above techniques for two illustrative cases: The case of a bar of square cross section and the case of a bar of circular cross section. A value of 0.3 was used for Poisson's ratio for all calculations. For the square cross section, the finite-difference solution previously described was used with an 11 by 11 grid as shown in figure 3. As a check on the accuracy of the finite-difference formulation, the results of the elastic solution which were obtained from the first iteration are

shown in the following table. These results are compared with those of reference 12.

	Reference 12	Finite differences
$\tau_{\max}/G\alpha a$	1.351	1.343
$M/G\alpha a^4$	2.250	2.244
$\varphi_{1,1}/G\alpha a^2$.589	.589

It is seen that the solution with this many grid points is sufficiently accurate.

The elastic-plastic calculations were performed assuming linear strain hardening. For linear strain hardening, equation (19f) can be written as

$$e_p = \frac{e_t - \frac{2}{3}(1 + \mu)}{1 + \frac{2}{3}(1 + \mu)\frac{m}{1 - m}} \quad (40)$$

where the strain-hardening parameter m is the ratio of the slope of the linear strain-hardening curve to the slope of the elastic curve. For the perfectly plastic case, m is equal to 0 and, for the elastic case, m is equal to 1. Figures 4 to 7 show the effects of the strain-hardening parameter and the angle of twist on the maximum stress,

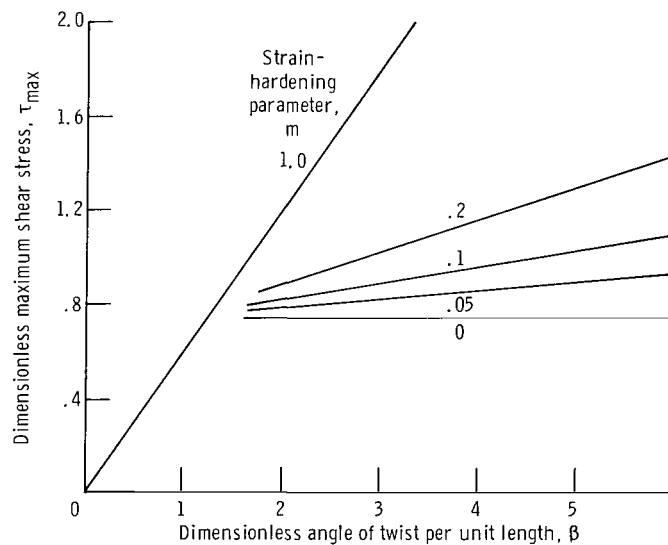


Figure 4. - Variation of dimensionless maximum shear stress with dimensionless angle of twist per unit length for several values of strain-hardening parameter for square cross section.

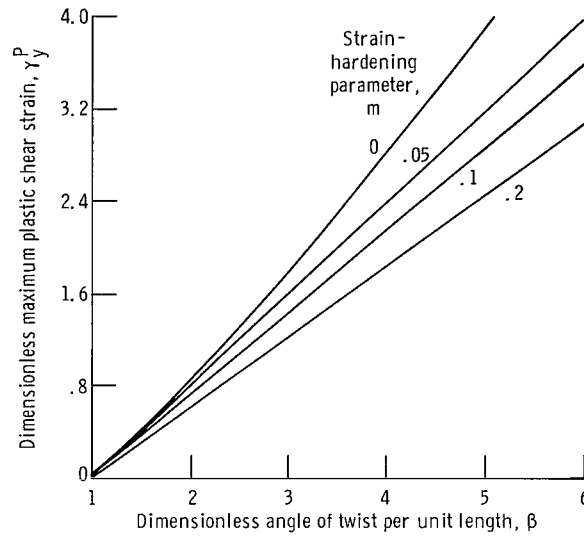


Figure 5. - Variation of dimensionless maximum plastic shear strain with dimensionless angle of twist per unit length for several values of strain-hardening parameter for square cross section.

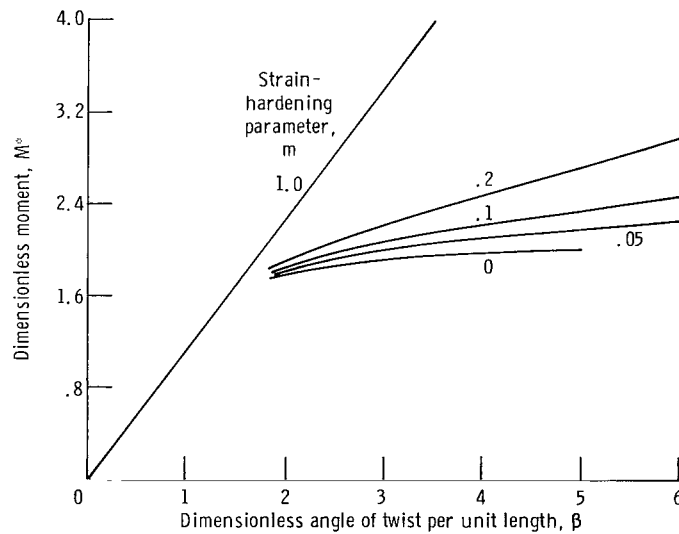


Figure 6. - Variation of dimensionless moment with dimensionless angle of twist per unit length for several values of the strain-hardening parameter for square cross section.

TABLE I. - SUMMARY OF RESULTS FOR TORSION OF
SQUARE PRISMATIC BAR

Linear strain-hardening parameter, m	Dimensionless angle of twist per unit length, β	Dimensionless torque, M^*	Maximum dimensionless shear stress, τ_{\max}	Maximum dimensionless strain, γ_{\max}^p
0	2	1.786	0.751	0.820
	3	1.918	↓	1.824
	4	1.955		2.851
	5	1.977		3.959
0.05	2	1.813	0.785	0.758
	3	1.997	.825	1.623
	4	2.094	.862	2.434
	5	2.166	.899	3.240
	6	2.228	.934	4.003
0.10	2	1.838	0.818	0.7015
	3	2.073	.893	1.478
	4	2.223	.963	2.209
	5	2.347	1.032	2.919
	6	2.465	1.099	3.618
0.20	2	1.890	0.881	0.600
	3	2.220	1.022	1.250
	4	2.471	1.156	1.870
	5	2.717	1.290	2.488
	6	2.966	1.426	3.116

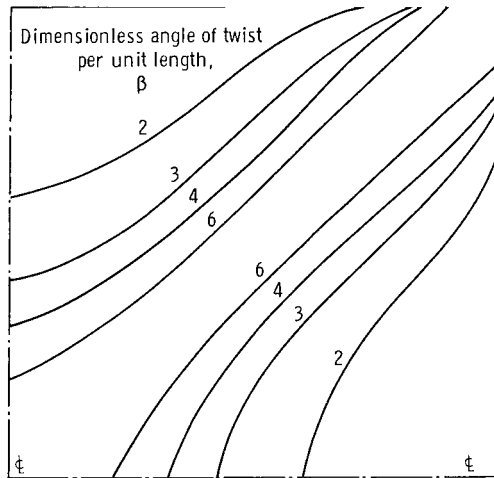


Figure 7. - Plastic zone boundaries in quadrant of square cross section as function of dimensionless angle of twist per unit length for strain-hardening parameter, 0.1.

the maximum plastic strain, the torque, and the size of the plastic zone. The results are also summarized in table I.

The results of the computations for a circular cross section are shown in figures 8 to 10. Linear strain hardening was assumed, and equations (32), (23), (24), (36), (26), and (38) were used. Figure 8 shows the elastic-plastic boundary as a function of β . Figure 9 shows the effect of the strain-hardening parameter on the shear stress for $\beta = 5.0$, and figure 10 shows the relation between these parameters and the torque. The corresponding quantities for the other values of m and β can easily be computed.

It is seen from the preceding discussion that solutions to the elastic-plastic torsion problem can be obtained in a relatively simple and straightforward manner. It is not necessary to assume that there is perfect plasticity or that the complete section has

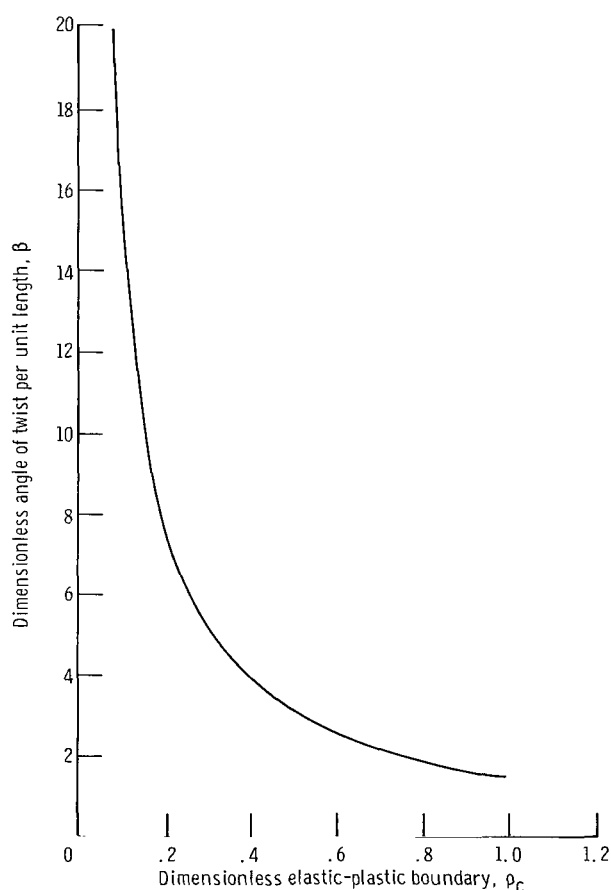


Figure 8. - Variation of dimensionless elastic-plastic boundary with dimensionless angle of twist per unit length for circular cross section.

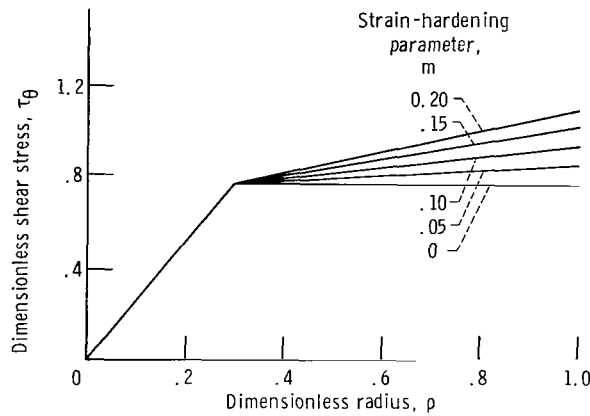


Figure 9. - Variation of dimensionless shear stress with dimensionless radius in circular cross section for various values of strain-hardening parameter. Dimensionless angle of twist per unit length, 5.0.

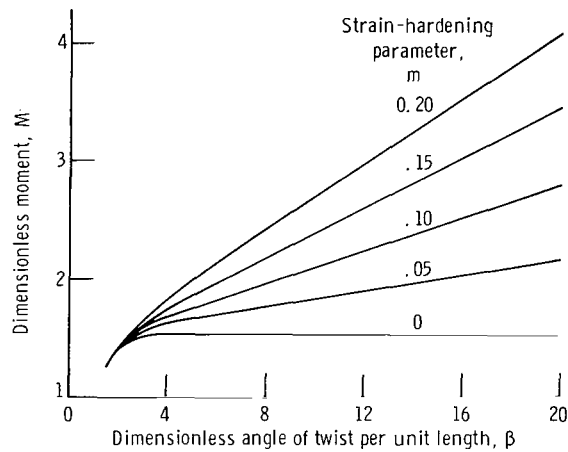


Figure 10. - Variation of dimensionless moment with dimensionless angle of twist per unit length for various values of strain-hardening parameter for circular cross section.

yielded. The elastic-plastic boundary is found automatically as part of the calculation and does not pose any special difficulties in its determination.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, October 4, 1967,
129-03-08-04-22.

APPENDIX A

SYMBOLS

A, B	constants	ϵ	strain
a	characteristic dimension of cross section	μ	Poisson's ratio
e	"dimensionless" strain	ξ, η	dimensionless coordinates, eq. (18)
G	shear modulus	ρ	dimensionless radius, eq. (28)
g	plastic strain function	σ	stress
M	torque acting on cross section	τ	resultant shear stress
M*	dimensionless torque, eq. (18)	$\tau_{xz}, \tau_{yz}, \tau_{\theta z}$	shear stresses
m	linear strain-hardening parameter	$\tau_x, \tau_y, \tau_\theta$	dimensionless shear stresses, eq. (18)
r	radial coordinate of circular cross section	φ	stress function
S _e	dimensionless equivalent stress	Subscripts:	
U	dimensionless stress function, eq. (18)	c	critical
u, v, w	displacements	e	equivalent
x, y, z	coordinates	i, j	station index
α	angle of twist per unit length	p	equivalent plastic
β	dimensionless angle of twist per unit length, eq. (18)	r	radial
$\gamma_{xz}, \gamma_{yz}, \gamma_{\theta z}$	engineering shear strains	t	equivalent total
$\gamma_x, \gamma_y, \gamma_z$	"dimensionless" shear strains, eq. (18)	θ	tangential
		0	yield
		Superscripts:	
		p	plastic

APPENDIX B

INCREMENTAL PLASTICITY EQUATIONS

If the incremental or flow theories of plasticity are to be used, the angle of twist per unit length α must be increased in small steps and the same calculation as previously described performed after each step. The stress-strain relations (eq. (7)) are written as follows:

$$\left. \begin{aligned} \gamma_{xz} &= \frac{1}{G} \tau_{xz} + \gamma_{xz}^p + \Delta\gamma_{xz}^p \\ \gamma_{yz} &= \frac{1}{G} \tau_{yz} + \gamma_{yz}^p + \Delta\gamma_{yz}^p \end{aligned} \right\} \quad (41)$$

where γ_{xz}^p and γ_{yz}^p are the accumulated plastic shear strains up to, but not including, the present increment of load, and $\Delta\gamma_{xz}^p$ and $\Delta\gamma_{yz}^p$ are the increments of plastic shear strains due to the current increment of load. The equation for the stress function φ now becomes

$$\nabla^2 \varphi = -2G(\alpha + \Delta\alpha) - g(x, y) - \Delta g \quad (42)$$

where $g(x, y)$ is as previously defined, and

$$\Delta g = G \left[\frac{\partial (\Delta\gamma_{xz}^p)}{\partial y} - \frac{\partial (\Delta\gamma_{yz}^p)}{\partial x} \right] \quad (43)$$

$g(x, y)$ is known, and Δg is to be determined. Equations (15) to (17) are replaced by the following:

$$\left. \begin{aligned} \Delta\gamma_{xz}^p &= \frac{\Delta\epsilon_p}{\epsilon_t} \gamma_{yz}^p \\ \Delta\gamma_{yz}^p &= \frac{\Delta\epsilon_p}{\epsilon_t} \gamma_{xz}^p \end{aligned} \right\} \quad (44)$$

where

$$\epsilon_t = \sqrt{\frac{(\gamma'_{xz})^2 + (\gamma'_{yz})^2}{3}} \quad (45)$$

$$\left. \begin{aligned} \gamma'_{xz} &= \gamma_{xz} - \gamma_{xz}^p \\ \gamma'_{yz} &= \gamma_{yz} - \gamma_{yz}^p \end{aligned} \right\} \quad (46)$$

$$\Delta\epsilon_p \cong \frac{\epsilon_t - \frac{2}{3} \frac{1+\mu}{E} \sigma_e}{1 + \frac{2}{3} \frac{1+\mu}{E} \frac{d\sigma_e}{d\epsilon_p}} \quad (47)$$

$$\sigma_e = \sqrt{3(\tau_{xz}^2 + \tau_{yz}^2)} \quad (48)$$

where σ_e , appearing in equation (47), is the value at the end of the previous increment of load and is known. Similarly, $d\sigma_e/d\epsilon_p$ is the slope obtained from the uniaxial stress-strain curve at the end of the previous increment of load and is known. Equation (47) is approximate except for linear strain hardening when it is exact and can be written as

$$\Delta\epsilon_p = \frac{\epsilon_t - \frac{2}{3} \frac{1+\mu}{E} \sigma_e}{1 + \frac{2}{3} (1+\mu) \frac{m}{1-m}} \quad (49)$$

The problem can now be solved for each increment of load by successive approximation as described in detail in references 8 and 9. For a given increment $\Delta\alpha$, values of $\Delta\gamma_{xz}^p$ and $\Delta\gamma_{yz}^p$ (such as zero) are assumed. Equation (42) is now solved for φ since Δg (as well as $g(x, y)$) is known. The stresses can be computed from equations (10), the strains from equation (41), the modified total strains from equations (46), ϵ_t from equation (45), $\Delta\epsilon_p$ from equation (47), and new values of $\Delta\gamma_{xz}^p$ and $\Delta\gamma_{yz}^p$ from equation (44). Then, Δg is recomputed and the process repeated until convergence is obtained.

By the above process, the plastic shear increments can readily be obtained for each increment of load. However, as previously pointed out, such an incremental process is not necessary for circular cross sections or any cross sections of perfectly plastic materials.

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